

# CONSTANT ANGLE SURFACES IN THE LORENTZIAN HEISENBERG GROUP

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**ABSTRACT.** In this paper, we define and, then, we characterize constant angle spacelike and timelike surfaces in the three-dimensional Heisenberg group, equipped with a 1-parameter family of Lorentzian metrics. In particular, we give an explicit local parametrization of these surfaces and we produce some examples.

## 1. INTRODUCTION

In recent years much work has been done to understand the geometry of surfaces whose unit normal vector field forms a constant angle with a fixed field of directions of the ambient space. These surfaces are called *helix surfaces* or *constant angle surfaces* and they have been studied in all the 3-dimensional geometries. Several classification results were obtained so far, in different ambient spaces and, among them, we mention [2, 4, 3, 5, 8, 10, 12, 13]. Moreover, helix submanifolds have been studied in higher dimensional euclidean spaces and product spaces in [7, 6, 14].

In the case of the Riemannian Bianchi-Cartan-Vranceanu (BCV) spaces  $\mathbb{E}^3(\kappa, \tau)$ , as they admit a Riemannian submersion onto a surface of constant Gaussian curvature (called the Hopf fibration), it was considered the angle  $\vartheta$  that the unit normal vector field of a surface in a BCV-space forms with the vector field tangent to the fibers of the Hopf fibration. This angle  $\vartheta$  has a crucial role in the study of surfaces in BCV-spaces as shown by Daniel, in [1], where he proved that the equations of Gauss and Codazzi are given in terms of the function  $\nu = \cos \vartheta$  and that this angle is one of the fundamental invariants for a surface in  $\mathbb{E}^3(\kappa, \tau)$ . Concerning the study of helix surfaces in the Lorentzian BCV-spaces  $\mathbb{L}^3(\kappa, \tau)$ , that are described by the 2-parameter family of Lorentzian metrics:

$$g_{\kappa, \tau} = \frac{dx^2 + dy^2}{F^2} - \left( dz - \tau \frac{y dx - x dy}{F} \right)^2, \quad F(x, y) = 1 + \frac{\kappa}{4}(x^2 + y^2), \quad \kappa, \tau \in \mathbb{R},$$

defined on  $\Omega \times \mathbb{R}$ , with  $\Omega = \{(x, y) \in \mathbb{R}^2 : F(x, y) > 0\}$ , we refer [9] and [11]. In [11], the authors classified constant angle spacelike surfaces in the Lorentz-Minkowski 3-space, while in [9] are considered constant angle spacelike and timelike surfaces in the Lorentzian product spaces given by  $\mathbb{S}^2 \times \mathbb{R}_1$  and  $\mathbb{H}^2 \times \mathbb{R}_1$ .

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We observe that the projection map  $\pi : \mathbb{L}^3(\kappa, \tau) \rightarrow M^2(\kappa)$ , given by  $\pi(x, y, z) = (x, y)$ , is a Riemannian submersion from  $\mathbb{L}^3(\kappa, \tau)$  to the surface of constant curvature  $\kappa$  given by

$$M^2(\kappa) = \left( \Omega, \frac{dx^2 + dy^2}{F^2} \right)$$

and, also, its fibers are the integral curves of the unit Killing vector field  $\partial_z$ , which is vertical with respect to  $\pi$ . The constant  $\tau$  is called the *bundle curvature parameter* of the ambient spaces  $\mathbb{L}^3(\kappa, \tau)$  and it satisfies the geometric identity:

$$(1) \quad \nabla^{k, \tau}_X \partial_z = \tau X \wedge \partial_z, \quad X \in \mathfrak{X}(\mathbb{L}^3(\kappa, \tau)),$$

where  $\nabla^{k, \tau}$  and  $\wedge$  denote, respectively, the Levi-Civita connection and the cross product of  $\mathbb{L}^3(\kappa, \tau)$ .

This paper is devoted to the study and the characterization of spacelike and timelike helix surfaces in the Lorentzian Heisenberg group given by  $\mathbb{L}^3(0, \tau)$  (with  $\tau \neq 0$ ), denoted by  $\mathbb{H}_3(\tau)$ , whose geometry we shall describe in Section 2. In Section 3 we determine the Gauss and Codazzi equations of an oriented pseudo-Riemannian surface  $\mathcal{M}$  immersed in  $\mathbb{H}_3(\tau)$ , proving that they involve the metric of  $\mathcal{M}$ , its shape operator  $A$ , the tangential projection  $T$  of the vertical vector field  $\partial_z$  and the function  $\nu := g_\tau(N, \partial_z) g_\tau(N, N)$ , where  $g_\tau := g_{0, \tau}$  and  $N$  is the unit normal to  $\mathcal{M}$ . Moreover, from the equation (1) derive two additional equations (see (12) and (13)) that are used to determine the shape operator and the Levi-Civita connection of  $\mathcal{M}$ .

In Sections 4 and 5 we define, respectively, the constant angle spacelike and timelike surfaces in  $\mathbb{H}_3(\tau)$  and we show that these surfaces have constant Gaussian curvature. Finally, in the Theorems 4.2 and 5.2 we establish the complete classification of these surfaces and, then, we construct some examples.

## 2. PRELIMINARIES

Let  $\mathbb{H}_3(\tau)$  (with  $\tau \neq 0$ ) denote the 3-dimensional Heisenberg group given by  $\mathbb{R}^3$  equipped with the 1-parameter family of Lorentzian metrics

$$g_\tau = dx^2 + dy^2 - (dz - \tau(y dx - x dy))^2,$$

which renders the map  $\pi : \mathbb{H}_3(\tau) \rightarrow \mathbb{R}^2$  a Riemannian submersion. With respect to this metric, the vector fields given by:

$$(2) \quad \begin{cases} E_1 = \frac{\partial}{\partial x} + \tau y \frac{\partial}{\partial z}, \\ E_2 = \frac{\partial}{\partial y} - \tau x \frac{\partial}{\partial z}, \\ E_3 = \frac{\partial}{\partial z}, \end{cases}$$

form a Lorentzian orthonormal basis on  $\mathbb{H}_3(\tau)$  and the associated Levi-Civita connection  $\nabla^\tau$ , where  $\nabla^\tau = \nabla^{0,\tau}$ , is given by:

$$(3) \quad \begin{aligned} \nabla^\tau_{E_1} E_1 &= \nabla^\tau_{E_2} E_2 = \nabla^\tau_{E_3} E_3 = 0, \\ \nabla^\tau_{E_2} E_1 &= \tau E_3 = -\nabla^\tau_{E_1} E_2, \\ \nabla^\tau_{E_3} E_1 &= -\tau E_2 = \nabla^\tau_{E_1} E_3, \\ \nabla^\tau_{E_3} E_2 &= \tau E_1 = \nabla^\tau_{E_2} E_3. \end{aligned}$$

We observe that  $E_3$  is a (timelike) unit Killing vector field, that is tangent to the fibers of the submersion  $\pi$  and it satisfies the following identity:

$$(4) \quad \nabla^\tau_X E_3 = \tau X \wedge E_3, \quad X \in \mathfrak{X}(\mathbb{H}_3(\tau)),$$

where  $\wedge$  is the cross product in  $\mathbb{H}_3(\tau)$  defined by the formula

$$U \wedge V = (u_2 v_3 - u_3 v_2) E_1 - (u_1 v_3 - u_3 v_1) E_2 - (u_1 v_2 - u_2 v_1) E_3.$$

Also, using the following convention

$$R^\tau(X, Y)Z = \nabla^\tau_X \nabla^\tau_Y Z - \nabla^\tau_Y \nabla^\tau_X Z - \nabla^\tau_{[X, Y]} Z,$$

the non zero components of the Riemann curvature tensor are:

$$(5) \quad \begin{aligned} R^\tau(E_1, E_2)E_1 &= -3\tau^2 E_2, & R^\tau(E_1, E_3)E_1 &= \tau^2 E_3, \\ R^\tau(E_1, E_2)E_2 &= 3\tau^2 E_1, & R^\tau(E_2, E_3)E_2 &= \tau^2 E_3, \\ R^\tau(E_2, E_3)E_3 &= \tau^2 E_2, & R^\tau(E_1, E_3)E_3 &= \tau^2 E_1. \end{aligned}$$

Moreover, the tensor  $R^\tau$  can be described as we have done in the following result.

**Proposition 2.1.** *The Riemann curvature tensor  $R^\tau$  of  $\mathbb{H}_3(\tau)$  is determined by*

$$(6) \quad \begin{aligned} R^\tau(X, Y)Z &= 3\tau^2 [g_\tau(Y, Z) X - g_\tau(X, Z) Y] \\ &+ 4\tau^2 [g_\tau(Y, E_3) g_\tau(Z, E_3) X - g_\tau(X, E_3) g_\tau(Z, E_3) Y \\ &+ g_\tau(Y, Z) g_\tau(X, E_3) E_3 - g_\tau(X, Z) g_\tau(Y, E_3) E_3], \end{aligned}$$

for all vector fields  $X, Y, Z$  on  $\mathbb{H}_3(\tau)$ .

*Proof.* Putting  $R^\tau(X \wedge Y, Z \wedge W) = g_\tau(R^\tau(X, Y)Z, W)$ , the matrix of  $R^\tau$  with respect to the basis  $\{E_2 \wedge E_3, E_3 \wedge E_1, E_1 \wedge E_2\}$  is given by:

$$R^\tau = \begin{pmatrix} -\tau^2 & 0 & 0 \\ 0 & -\tau^2 & 0 \\ 0 & 0 & -3\tau^2 \end{pmatrix}.$$

Now, we set  $X = \bar{X} + x E_3$ , where  $\bar{X}$  is the horizontal component of  $X$  and  $x = -g_\tau(X, E_3)$ , etc. Therefore, we obtain that

$$\begin{aligned} g_\tau(R^\tau(X, Y)Z, W) &= g_\tau(R^\tau(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) \\ &+ y z g_\tau(R^\tau(\bar{X}, E_3)E_3, \bar{W}) + x z g_\tau(R^\tau(E_3, \bar{Y})E_3, \bar{W}) \\ &+ w x g_\tau(R^\tau(E_3, \bar{Y})\bar{Z}, E_3) + y w g_\tau(R^\tau(\bar{X}, E_3)\bar{Z}, E_3), \end{aligned}$$

where it's easy to see that

$$g_\tau(R^\tau(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) = 3\tau^2 [g_\tau(\bar{X}, \bar{W}) g_\tau(\bar{Y}, \bar{Z}) - g_\tau(\bar{X}, \bar{Z}) g_\tau(\bar{Y}, \bar{W})].$$

Also, as

$$R^\tau(\bar{X}, E_3) E_3 = \tau^2 \bar{X}, \quad R^\tau(E_3, \bar{Y}) E_3 = -\tau^2 \bar{Y},$$

it results that

$$\begin{aligned} g_\tau(R^\tau(X, Y) Z, W) &= 3\tau^2 [g_\tau(\bar{X}, \bar{W}) g_\tau(\bar{Y}, \bar{Z}) - g_\tau(\bar{X}, \bar{Z}) g_\tau(\bar{Y}, \bar{W})] \\ &\quad + \tau^2 [z y g_\tau(\bar{X}, \bar{W}) + x w g_\tau(\bar{Y}, \bar{Z}) - x z g_\tau(\bar{Y}, \bar{W}) - w y g_\tau(\bar{X}, \bar{Z})] \\ &= 3\tau^2 [g_\tau(Y, Z) g_\tau(X, W) - g_\tau(X, Z) g_\tau(Y, W)] \\ &\quad + 4\tau^2 [g_\tau(Y, E_3) g_\tau(Z, E_3) g_\tau(X, W) - g_\tau(X, E_3) g_\tau(Z, E_3) g_\tau(Y, W) \\ &\quad + g_\tau(Y, Z) g_\tau(X, E_3) g_\tau(E_3, W) - g_\tau(X, Z) g_\tau(Y, E_3) g_\tau(E_3, W)]. \end{aligned}$$

Since  $W$  is arbitrary, we obtain the equation (6).  $\square$

### 3. THE STRUCTURE EQUATIONS FOR SURFACES IN $\mathbb{H}_3(\tau)$

In this section, we will determine the structure equations for a surface  $\mathcal{M}$  immersed into the Lorentzian Heisenberg group  $\mathbb{H}_3(\tau)$  that will be used, in the following sections, to study the constant angle surfaces in this ambient space. We remember that a surface  $\mathcal{M}$  is called *space-like* (respectively, *timelike*) if the induced metric on  $\mathcal{M}$  via the immersion is a Riemannian (respectively, Lorentzian) metric.

Let  $\mathcal{M}$  be an oriented pseudo-Riemannian surface immersed into  $\mathbb{H}_3(\tau)$  and let  $N$  be a unit normal vector field, that is  $g_\tau(N, N) = \varepsilon$ , where  $\varepsilon = -1$  (respectively,  $\varepsilon = 1$ ) if  $\mathcal{M}$  is a spacelike (respectively, a timelike) surface.

The Gauss and Weingarten formulas, for all  $X, Y \in C(T\mathcal{M})$ , are

$$\begin{aligned} (7) \quad \nabla^\tau_X Y &= \nabla_X Y + \alpha(X, Y), \\ \nabla^\tau_X N &= -A(X), \end{aligned}$$

where with  $A$  we have indicated the shape operator of  $\mathcal{M}$  in  $\mathbb{H}_3(\tau)$ , with  $\nabla$  the induced Levi-Civita connection on  $\mathcal{M}$  and by  $\alpha$  the second fundamental form of  $\mathcal{M}$  in  $\mathbb{H}_3(\tau)$ . Note that the second fundamental form can be written as

$$\alpha(X, Y) = \varepsilon g_\tau(A(X), Y) N, \quad X, Y \in C(T\mathcal{M}).$$

If we project the vector field  $E_3$  onto the tangent plane to  $\mathcal{M}$ , we have

$$E_3 = T + \nu N,$$

for a certain smooth function  $\nu$  defined on  $\mathcal{M}$ . Here  $T$  is the tangent part of  $E_3$  which satisfies

$$(8) \quad g_\tau(T, T) = -(1 + \varepsilon \nu^2).$$

We observe that, for all  $X \in T\mathcal{M}$ , we have that

$$\begin{aligned} (9) \quad \nabla^\tau_X E_3 &= \nabla^\tau_X T + X(\nu) N + \nu \nabla^\tau_X N \\ &= \nabla_X T + X(\nu) N + \varepsilon g_\tau(A(X), T) N - \nu A(X). \end{aligned}$$

On the other hand, writing  $X = \sum_{i=1}^3 X_i E_i$  and using (4), we get:

$$\begin{aligned} (10) \quad \nabla^\tau_X E_3 &= \tau X \wedge E_3 \\ &= \varepsilon \tau g_\tau(JX, T) N - \tau \nu JX, \end{aligned}$$



where  $JX = N \wedge X$  denotes the rotation of angle  $\pi/2$  on  $T\mathcal{M}$  and it satisfies

$$(11) \quad g_\tau(JX, JX) = -\varepsilon g_\tau(X, X), \quad J^2X = \varepsilon X.$$

Identifying the tangent and normal components of (9) and (10) respectively, we obtain

$$(12) \quad \nabla_X T = \nu (A(X) - \tau JX)$$

and

$$(13) \quad X(\nu) = -\varepsilon g_\tau(A(X) - \tau JX, T).$$

In the following result, we will give the expression of the Gauss and Codazzi equations for a pseudo-Riemannian surface  $\mathcal{M}$  in  $\mathbb{H}_3(\tau)$ .

**Proposition 3.1.** *Under the previous conditions, the Gauss and Codazzi equations in  $\mathbb{H}_3(\tau)$  are given, respectively, by:*

$$(14) \quad K = \overline{K} + \varepsilon \det A = \varepsilon (\det A - 4\tau^2 \nu^2) - \tau^2$$

and

$$(15) \quad \nabla_X A(Y) - \nabla_Y A(X) - A[X, Y] = -4\varepsilon \nu \tau^2 [g_\tau(Y, T)X - g_\tau(X, T)Y],$$

where  $X$  and  $Y$  are tangent vector fields on  $\mathcal{M}$ ,  $K$  is the Gauss curvature of  $\mathcal{M}$  and  $\overline{K}$  denotes the sectional curvature in  $\mathbb{H}_3(\tau)$  of the plane tangent to  $\mathcal{M}$ .

*Proof.* We start proving the equation (14). Using that

$$\begin{aligned} & g_\tau(\alpha(X, X), \alpha(Y, Y)) - g_\tau(\alpha(X, Y), \alpha(X, Y))^2 \\ &= \varepsilon [g_\tau(A(X), X) g_\tau(A(Y), Y) - g_\tau(A(X), Y)^2], \end{aligned}$$

the Gauss equation can be written as

$$(16) \quad K = \overline{K} + \varepsilon \frac{g_\tau(A(X), X)g_\tau(A(Y), Y) - g_\tau(A(X), Y)^2}{g_\tau(X, X)g_\tau(Y, Y) - g_\tau(X, Y)^2}.$$

If we suppose that  $\{X, Y\}$  is a local orthonormal frame on  $\mathcal{M}$ , i.e.  $g_\tau(X, X) = 1$ ,  $g_\tau(X, Y) = 0$ ,  $g_\tau(Y, Y) = -\varepsilon$ , we get

$$g_\tau(A(X), X) g_\tau(A(Y), Y) - g_\tau(A(X), Y)^2 = -\varepsilon \det A.$$

Also, from (6), we obtain that

$$(17) \quad \begin{aligned} -\varepsilon \overline{K} &= g_\tau(R^\tau(X, Y)Y, X) = -3\varepsilon \tau^2 + 4\tau^2 [g_\tau(Y, T)^2 - \varepsilon g_\tau(X, T)^2] \\ &= -\varepsilon \tau^2 [3 + 4g_\tau(T, T)]. \end{aligned}$$

Now, substituting (17) in (16), and using (8), we have the equation (14).

To obtain (15), we start from the Codazzi equation for hypersurfaces that is given by:

$$g_\tau(R^\tau(X, Y)Z, N) = g_\tau(\nabla_X A(Y) - \nabla_Y A(X) - A[X, Y], Z).$$

Also, from Proposition 2.1 we get

$$\begin{aligned} R^\tau(X, Y)N &= 4\tau^2 g_\tau(N, E_3) [g_\tau(Y, E_3)X - g_\tau(X, E_3)Y] \\ &= 4\varepsilon \nu \tau^2 [g_\tau(Y, T)X - g_\tau(X, T)Y]. \end{aligned}$$

Therefore, we obtain (15). □

Now, we are ready to begin the study of the constant angle surfaces in  $\mathbb{H}_3(\tau)$ . Firstly, we give the following:

**Definition 3.2.** Let  $\mathcal{M}$  be an oriented pseudo-Riemannian surface in the Lorentzian Heisenberg group  $\mathbb{H}_3(\tau)$  and let  $N$  be a unit normal vector field, with  $g_\tau(N, N) = \varepsilon$ . We say that  $\mathcal{M}$  is a *helix surface* or a *constant angle surface* if the function  $\nu := \varepsilon g_\tau(N, E_3)$  is constant at every point of the surface.

#### 4. CONSTANT ANGLE SPACELIKE SURFACES

Let  $\mathcal{M}$  be a spacelike surface in  $\mathbb{H}_3(\tau)$ . As  $\varepsilon = -1$ , from the equation (8) it follows that (up to the orientation of  $N$ ) we can write  $\nu = \cosh \vartheta$ , where  $\vartheta \geq 0$  is called *the hyperbolic angle function* between  $N$  and  $E_3$ .

From now on, we assume that the function  $\vartheta$  is constant. Note that  $\vartheta \neq 0$ . In fact, if it were then the vector fields  $E_1$  and  $E_2$  would be tangent to the surface  $\mathcal{M}$ , which is absurd since as  $\tau \neq 0$  the horizontal distribution of the submersion  $\pi$  is not integrable.

**Lemma 4.1.** *Let  $\mathcal{M}$  be a helix spacelike surface in  $\mathbb{H}_3(\tau)$  with constant angle  $\vartheta > 0$ . Then, we have the following properties.*

- (i) *With respect to the basis  $\{T, JT\}$ , the matrix associated to the shape operator  $A$  takes the form*

$$A = \begin{pmatrix} 0 & -\tau \\ -\tau & \lambda \end{pmatrix},$$

*for some smooth function  $\lambda$  on  $\mathcal{M}$ .*

- (ii) *The Levi-Civita connection  $\nabla$  of  $\mathcal{M}$  is given by*

$$\begin{aligned} \nabla_T T &= -2\tau \cosh \vartheta JT, & \nabla_{JT} T &= \lambda \cosh \vartheta JT, \\ \nabla_T JT &= 2\tau \cosh \vartheta T, & \nabla_{JT} JT &= -\lambda \cosh \vartheta T. \end{aligned}$$

- (iii) *The Gauss curvature of  $\mathcal{M}$  is constant and satisfies*

$$K = 4\tau^2 \cosh^2 \vartheta.$$

- (iv) *The function  $\lambda$  satisfies the equation*

$$(18) \quad T(\lambda) + \lambda^2 \cosh \vartheta + 4\tau^2 \cosh^3 \vartheta = 0.$$

*Proof.* Point (i) follows directly from (13). From (12) and using

$$g_\tau(T, T) = g_\tau(JT, JT) = \sinh^2 \vartheta, \quad g_\tau(T, JT) = 0,$$

we obtain (ii). From the Gauss equation (14) in  $\mathbb{H}_3(\tau)$ , we have that the Gauss curvature of  $\mathcal{M}$  is given by

$$K = 4\tau^2 \nu^2 - (\det A + \tau^2) = 4\tau^2 \cosh^2 \vartheta.$$

Finally, equation (18) follows from the Codazzi equation (15) putting  $X = T$ ,  $Y = JT$  and using (ii). In fact, it is easy to see that

$$4\tau^2 \cosh \vartheta [g_\tau(JT, T)T - g_\tau(T, T)JT] = -4\tau^2 \cosh \vartheta \sinh^2 \vartheta JT$$

and

$$\begin{aligned} & \nabla_T A(JT) - \nabla_{JT} A(T) - A[T, JT] \\ &= \nabla_T(-\tau T + \lambda JT) - \nabla_{JT}(-\tau JT) - A(2\tau \cosh \vartheta T - \lambda \cosh \vartheta JT) \\ &= (4\tau^2 \cosh \vartheta + T(\lambda) + \lambda^2 \cosh \vartheta) JT. \end{aligned}$$

□

From  $g_\tau(E_3, N) = -\cosh \vartheta$ , it follows that there exists a smooth function  $\varphi$  on  $\mathcal{M}$  such that

$$N = \sinh \vartheta \cos \varphi E_1 + \sinh \vartheta \sin \varphi E_2 + \cosh \vartheta E_3.$$

Therefore, we can write

$$(19) \quad T = E_3 - \cosh \vartheta N = -\sinh \vartheta [\cosh \vartheta \cos \varphi E_1 + \cosh \vartheta \sin \varphi E_2 + \sinh \vartheta E_3]$$

and

$$(20) \quad JT = \sinh \vartheta (\sin \varphi E_1 - \cos \varphi E_2).$$

Moreover, we have

$$(21) \quad \begin{cases} A(T) = -\nabla^\tau_T N = [T(\varphi) + \tau \cosh^2 \vartheta + \tau \sinh^2 \vartheta] JT, \\ A(JT) = -\nabla^\tau_{JT} N = (JT)(\varphi) JT - \tau T. \end{cases}$$

Comparing (21) with (i) of Lemma 4.1, it results that

$$(22) \quad \begin{cases} (JT)(\varphi) = \lambda, \\ T(\varphi) = -2\tau \cosh^2 \vartheta. \end{cases}$$

Also, as

$$[T, JT] = \cosh \vartheta (2\tau T - \lambda JT),$$

the compatibility condition of system (22):

$$(\nabla_T JT - \nabla_{JT} T)(\varphi) = [T, JT](\varphi) = T(JT(\varphi)) - JT(T(\varphi))$$

is equivalent to (18).

We now choose local coordinates  $(u, v)$  on  $\mathcal{M}$  such that

$$(23) \quad \partial_u = T.$$

Also, as  $\partial_v$  is tangent to  $\mathcal{M}$ , it can be written in the form

$$(24) \quad \partial_v = aT + bJT,$$

for certain functions  $a = a(u, v)$  and  $b = b(u, v)$ . As

$$0 = [\partial_u, \partial_v] = (a_u + 2\tau b \cosh \vartheta) T + (b_u - b \lambda \cosh \vartheta) JT,$$

then

$$(25) \quad \begin{cases} a_u = -2\tau b \cosh \vartheta, \\ b_u = b \lambda \cosh \vartheta. \end{cases}$$

Moreover, the equation (18) of Lemma 4.1 can be written as

$$(26) \quad \lambda_u + \cosh \vartheta \lambda^2 + 4\tau^2 \cosh^3 \vartheta = 0.$$

Integrating (26), we obtain that

$$(27) \quad \lambda(u, v) = 2\tau \cosh \vartheta \tan[\eta(v) - 2\tau(\cosh \vartheta)^2 u],$$

for some smooth function  $\eta$  depending on  $v$  and we can solve system (25). Remark that we are interested in only one coordinate system on the surface  $\mathcal{M}$  and, hence, we only need one solution for  $a$  and  $b$ , for example:

$$\begin{cases} a(u, v) = \frac{\sin(\eta(v) - 2\tau(\cosh \vartheta)^2 u)}{\cosh \vartheta}, \\ b(u, v) = \cos(\eta(v) - 2\tau(\cosh \vartheta)^2 u). \end{cases}$$

Moreover, using these expressions, we have that the system (22) becomes

$$\begin{cases} \varphi_u = -2\tau \cosh^2 \vartheta, \\ \varphi_v = 0, \end{cases}$$

of which the general solution is given by

$$(28) \quad \varphi(u, v) = -2\tau \cosh^2 \vartheta u + c,$$

where  $c$  is a real constant.

With respect to the local coordinates  $(u, v)$  chosen above, we have the following characterization of the position vector of a helix spacelike surface.

**Theorem 4.2.** *Let  $\mathcal{M}$  be a helix spacelike surface in  $\mathbb{H}_3(\tau)$  with constant angle  $\vartheta > 0$ . Then, with respect to the local coordinates  $(u, v)$  on  $\mathcal{M}$  defined in (23) and (24), the position vector  $F$  of  $\mathcal{M}$  in  $\mathbb{R}^3$  is given by*

$$(29) \quad \begin{aligned} F(u, v) = & \left( \frac{\tanh \vartheta}{2\tau} \sin u + f_1(v), -\frac{\tanh \vartheta}{2\tau} \cos u + f_2(v), \right. \\ & \left. -\frac{(\sinh \vartheta)^2}{2} u + \frac{\tanh \vartheta}{2} [f_1(v) \cos u + f_2(v) \sin u] + f_3(v) \right), \end{aligned}$$

with

$$f_1'(v)^2 + f_2'(v)^2 = (\sinh \vartheta)^2, \quad f_3'(v) = \tau (f_2(v) f_1'(v) - f_1(v) f_2'(v)).$$

*Proof.* Let  $\mathcal{M}$  be a helix spacelike surface in  $\mathbb{H}_3(\tau)$  with constant angle  $\vartheta \in (0, +\infty)$  and let  $F$  be the position vector of  $\mathcal{M}$  in  $\mathbb{R}^3$ . Then, with respect to the local coordinates  $(u, v)$  on  $\mathcal{M}$  defined in (23) and (24), we can write  $F(u, v) = (F_1(u, v), F_2(u, v), F_3(u, v))$ , with  $(u, v) \in \Omega \subset \mathbb{R}^2$ . By definition, taking into account (19) and (20), we have that

$$\begin{aligned} \partial_u F &= (\partial_u F_1, \partial_u F_2, \partial_u F_3) = T \\ &= -\sinh \vartheta [\cosh \vartheta \cos \varphi E_{1|F(u,v)} + \cosh \vartheta \sin \varphi E_{2|F(u,v)} + \sinh \vartheta E_{3|F(u,v)}] \end{aligned}$$

and

$$\begin{aligned} \partial_v F &= (\partial_v F_1, \partial_v F_2, \partial_v F_3) = aT + bJT \\ &= \sinh \vartheta [(-a \cosh \vartheta \cos \varphi + b \sin \varphi) E_{1|F(u,v)} \\ &\quad - (a \cosh \vartheta \sin \varphi + b \cos \varphi) E_{2|F(u,v)} - a \sinh \vartheta E_{3|F(u,v)}]. \end{aligned}$$

Therefore, using the expression of  $E_1$ ,  $E_2$  and  $E_3$  with respect to the coordinates vector fields of  $\mathbb{R}^3$ , it results that

$$(30) \quad \begin{cases} \partial_u F_1 = -\sinh \vartheta \cosh \vartheta \cos \varphi, \\ \partial_u F_2 = -\sinh \vartheta \cosh \vartheta \sin \varphi, \\ \partial_u F_3 = -\sinh \vartheta (\tau \cosh \vartheta \cos \varphi F_2 - \tau \cosh \vartheta \sin \varphi F_1 + \sinh \vartheta) \end{cases}$$

and

$$(31) \quad \begin{cases} \partial_v F_1 = \sinh \vartheta (-a \cosh \vartheta \cos \varphi + b \sin \varphi), \\ \partial_v F_2 = -\sinh \vartheta (a \cosh \vartheta \sin \varphi + b \cos \varphi), \\ \partial_v F_3 = \sinh \vartheta [\tau(-a \cosh \vartheta \cos \varphi + b \sin \varphi) F_2 \\ \quad + \tau(a \cosh \vartheta \sin \varphi + b \cos \varphi) F_1 - a \sinh \vartheta]. \end{cases}$$

From the first two equations of (30), we obtain that

$$\begin{cases} F_1(u, v) = \frac{\tanh \vartheta}{2\tau} \sin \varphi(u) + f_1(v), \\ F_2(u, v) = -\frac{\tanh \vartheta}{2\tau} \cos \varphi(u) + f_2(v). \end{cases}$$

Then, using these expressions in the third equation of (30) and integrating, we get

$$F_3(u, v) = -\frac{(\sinh \vartheta)^2 u}{2} + \frac{\tanh \vartheta}{2} (f_1(v) \cos \varphi(u) + f_2(v) \sin \varphi(u)) + f_3(v).$$

Consequently, from (31), we have that

$$(32) \quad \begin{cases} f_1'(v) = -\sinh \vartheta \sin(\eta(v) - c), \\ f_2'(v) = -\sinh \vartheta \cos(\eta(v) - c), \\ f_3'(v) = \tau (f_2(v) f_1'(v) - f_1(v) f_2'(v)). \end{cases}$$

Using the change of variable  $\varphi(u) \mapsto u$ , we obtain the parametrization given in (29).  $\square$

Now, we present some examples of constant angle spacelike surfaces in  $\mathbb{H}_3(\tau)$  obtained using the parametrization given in the Theorem 4.2.

**Example 4.3.** Choosing  $\eta(v) = v + c$  in (32), we get

$$\begin{cases} f_1(v) = \sinh \vartheta \cos v, \\ f_2(v) = -\sinh \vartheta \sin v, \\ f_3(v) = \tau v \sinh^2 \vartheta. \end{cases}$$

Substituting these expressions in (29) we have explicit parametrizations of helix spacelike surfaces that depend only of the hyperbolic angle  $\vartheta$ .

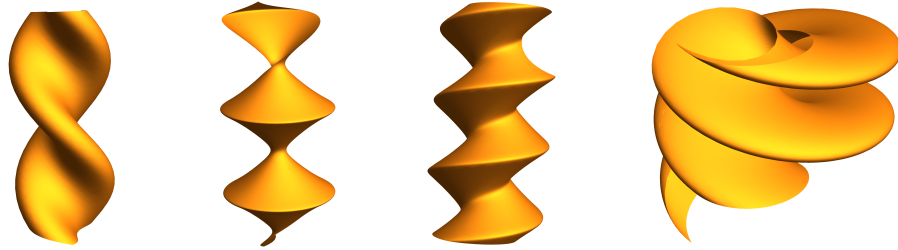


FIGURE 1. Constant angle spacelike surfaces for  $\vartheta = \pi/3$ ,  $\vartheta = \pi/4$ ,  $\vartheta = \pi/6$  and  $\vartheta = \pi/8$ .

## 5. CONSTANT ANGLE TIMELIKE SURFACES

Now we are going to study, following the same procedure as in Section 4, the constant angle timelike surfaces in  $\mathbb{H}_3(\tau)$ .

Let  $\mathcal{M}$  be a timelike surface in the Lorentzian Heisenberg group  $\mathbb{H}_3(\tau)$ . As  $\varepsilon = 1$ , from the equation (8) it follows that (up to the orientation of  $N$ ) we can write  $\nu = \sinh \vartheta$ , where  $\vartheta$  is the *hyperbolic angle function* between  $N$  and  $E_3$ . We observe that, from (11), it results that

$$g_\tau(JT, JT) = -g_\tau(T, T) = \cosh^2 \vartheta.$$

We suppose that the function  $\vartheta$  is constant and we observe that if  $\vartheta = 0$ , we have that  $E_3$  is always tangent to  $\mathcal{M}$  and, therefore,  $\mathcal{M}$  is a Hopf cylinder. Therefore, from now on we assume that the constant angle  $\vartheta \neq 0$ .

**Lemma 5.1.** *Let  $\mathcal{M}$  be a helix timelike surface in  $\mathbb{H}_3(\tau)$  with constant angle  $\vartheta \neq 0$ . Then, we have the following properties.*

- (i) *With respect to the basis  $\{T, JT\}$ , the matrix associated to the shape operator  $A$  takes the form*

$$A = \begin{pmatrix} 0 & \tau \\ -\tau & \lambda \end{pmatrix},$$

*for some smooth function  $\lambda$  on  $\mathcal{M}$ .*

- (ii) *The Levi-Civita connection  $\nabla$  of  $\mathcal{M}$  is given by*

$$\begin{aligned} \nabla_T T &= -2\tau \sinh \vartheta JT, & \nabla_{JT} T &= \lambda \sinh \vartheta JT, \\ \nabla_T JT &= -2\tau \sinh \vartheta T, & \nabla_{JT} JT &= \lambda \sinh \vartheta T. \end{aligned}$$

- (iii) *The Gauss curvature of  $\mathcal{M}$  is the constant given by:*

$$K = -4\tau^2 \sinh^2 \vartheta.$$

- (iv) *The function  $\lambda$  satisfies the equation*

$$(33) \quad T(\lambda) + \lambda^2 \sinh \vartheta + 4\tau^2 \sinh^3 \vartheta = 0.$$

*Proof.* The proof is analogous to that of Lemma 4.1, taking into account that  $T$  is timelike,  $JT$  is spacelike and that the operator of rotation  $J$  satisfies  $J^2 = I$  (see (11)).  $\square$

From  $g_\tau(E_3, N) = \sinh \vartheta$ , it follows that there exists a smooth function  $\varphi$  on  $\mathcal{M}$  such that

$$N = \cosh \vartheta \cos \varphi E_1 + \cosh \vartheta \sin \varphi E_2 - \sinh \vartheta E_3.$$

Consequently, we obtain that

$$(34) \quad \begin{cases} T = E_3 - \sinh \vartheta N = \cosh \vartheta [-\sinh \vartheta \cos \varphi E_1 - \sinh \vartheta \sin \varphi E_2 + \cosh \vartheta E_3], \\ JT = \cosh \vartheta (\sin \varphi E_1 - \cos \varphi E_2). \end{cases}$$

In this case, we have that

$$(35) \quad \begin{aligned} A(T) &= -\nabla^\tau_T N = [T(\varphi) - \tau \cosh^2 \vartheta - \tau \sinh^2 \vartheta] JT, \\ A(JT) &= -\nabla^\tau_{JT} N = (JT)(\varphi) JT + \tau T. \end{aligned}$$

Comparing (35) with (i) of Lemma 5.1, we get

$$(36) \quad \begin{cases} (JT)(\varphi) = \lambda, \\ T(\varphi) = 2\tau \sinh^2 \vartheta. \end{cases}$$

Also, as

$$[T, JT] = -\sinh \vartheta (2\tau T + \lambda JT),$$

the compatibility condition of system (36) is given by:

$$(\nabla_T JT - \nabla_{JT} T)(\varphi) = [T, JT](\varphi) = T(JT(\varphi)) - JT(T(\varphi))$$

and it is equivalent to the equation (33).

Now, we choose local coordinates  $(u, v)$  on  $\mathcal{M}$  such that

$$(37) \quad \partial_u = T, \quad \partial_v = aT + bJT,$$

for certain functions  $a = a(u, v)$  and  $b = b(u, v)$ . As

$$0 = [\partial_u, \partial_v] = (a_u - 2\tau b \sinh \vartheta) T + (b_u - b \lambda \sinh \vartheta) JT,$$

then

$$(38) \quad \begin{cases} a_u = 2\tau b \sinh \vartheta, \\ b_u = b \lambda \sinh \vartheta. \end{cases}$$

Moreover, the equation (33) of Lemma 5.1 can be written as

$$\lambda_u + \sinh \vartheta \lambda^2 + 4\tau^2 \sinh^3 \vartheta = 0$$

and, solving this equation, one finds

$$\lambda(u, v) = 2\tau \sinh \vartheta \tan[\eta(v) - 2\tau(\sinh \vartheta)^2 u],$$

for some smooth function  $\eta$  depending on  $v$ . As we are interested in only one coordinate system on the surface  $\mathcal{M}$ , we can consider the following solution of the system (38):

$$\begin{cases} a(u, v) = -\frac{\sin(\eta(v) - 2\tau(\sinh \vartheta)^2 u)}{\sinh \vartheta}, \\ b(u, v) = \cos(\eta(v) - 2\tau(\sinh \vartheta)^2 u). \end{cases}$$

Also, using these expressions, we have that the general solution of the system (36) is given by:

$$\varphi(u, v) = 2\tau(\sinh \vartheta)^2 u + c, \quad c \in \mathbb{R}.$$

**Theorem 5.2.** *Let  $\mathcal{M}$  be a helix timelike surface in  $\mathbb{H}_3(\tau)$  with constant angle  $\vartheta \neq 0$ . Then, with respect to the local coordinates  $(u, v)$  on  $\mathcal{M}$  defined in (37) the position vector  $F$  of  $\mathcal{M}$  in  $\mathbb{R}^3$  is given by:*

$$(39) \quad F(u, v) = \left( -\frac{\coth \vartheta}{2\tau} \sin u + f_1(v), \frac{\coth \vartheta}{2\tau} \cos u + f_2(v), \frac{(\cosh \vartheta)^2}{2} u - \frac{\coth \vartheta}{2} [f_1(v) \cos u + f_2(v) \sin u] + f_3(v) \right),$$

with

$$f_1'(v)^2 + f_2'(v)^2 = (\cosh \vartheta)^2, \quad f_3'(v) = \tau (f_2(v) f_1'(v) - f_1(v) f_2'(v)).$$

*Proof.* With respect to the local coordinates  $(u, v)$  on the helix timelike surface  $\mathcal{M}$ , given in (37), we can parametrize the surface as

$$F(u, v) = (F_1(u, v), F_2(u, v), F_3(u, v)), \quad (u, v) \in \Omega \subset \mathbb{R}^2.$$

From the expressions (34), it results that

$$(40) \quad \begin{cases} \partial_u F_1 = -\sinh \vartheta \cosh \vartheta \cos \varphi, \\ \partial_u F_2 = -\sinh \vartheta \cosh \vartheta \sin \varphi, \\ \partial_u F_3 = \cosh \vartheta (-\tau \sinh \vartheta \cos \varphi F_2 + \tau \sinh \vartheta \sin \varphi F_1 + \cosh \vartheta) \end{cases}$$

and

$$(41) \quad \begin{cases} \partial_v F_1 = \cosh \vartheta (-a \sinh \vartheta \cos \varphi + b \sin \varphi), \\ \partial_v F_2 = -\cosh \vartheta (a \sinh \vartheta \sin \varphi + b \cos \varphi), \\ \partial_v F_3 = \cosh \vartheta [\tau (-a \sinh \vartheta \cos \varphi + b \sin \varphi) F_2 \\ + \tau (a \sinh \vartheta \sin \varphi + b \cos \varphi) F_1 + a \cosh \vartheta]. \end{cases}$$

Integrating (40), we obtain that

$$\begin{cases} F_1(u, v) = -\frac{\cosh \vartheta}{2\tau} \sin \varphi(u) + f_1(v), \\ F_2(u, v) = \frac{\cosh \vartheta}{2\tau} \cos \varphi(u) + f_2(v), \\ F_3(u, v) = \frac{(\cosh \vartheta)^2 u}{2} - \frac{\cosh \vartheta}{2} (f_1(v) \cos \varphi(u) + f_2(v) \sin \varphi(u)) + f_3(v), \end{cases}$$

where, from (41), the functions  $f_i(v)$ ,  $i = 1, 2, 3$ , satisfy the following relations:

$$(42) \quad \begin{cases} f_1'(v) = \cosh \vartheta \sin(\eta(v) + c), \\ f_2'(v) = -\cosh \vartheta \cos(\eta(v) + c), \\ f_3'(v) = \tau (f_2(v) f_1'(v) - f_1(v) f_2'(v)). \end{cases}$$

Finally, using the change of variable  $\varphi(u) \mapsto u$ , we obtain the parametrization of  $\mathcal{M}$  given in (39).  $\square$

We end the section giving some examples of constant angle timelike surfaces in  $\mathbb{H}_3(\tau)$  constructed from the parametrization obtained in the Theorem 5.2.

**Example 5.3.** If we choose  $\eta(v) = v - c$  in (42), we have the expressions:

$$\begin{cases} f_1(v) = -\cosh \vartheta \cos v, \\ f_2(v) = -\cosh \vartheta \sin v, \\ f_3(v) = -\tau v \cosh^2 \vartheta \end{cases}$$

and, using (39), we obtain explicit parametrizations of helix timelike surfaces that depend only of the hyperbolic angle  $\vartheta$ .



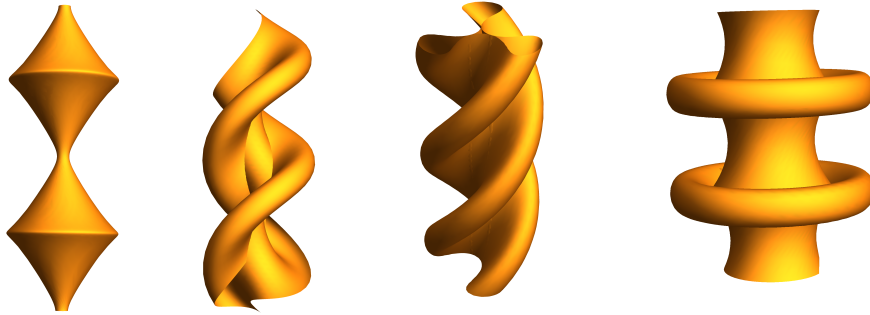


FIGURE 2. *Constant angle timelike surfaces for  $\vartheta = \pi/3$ ,  $\vartheta = \pi/4$ ,  $\vartheta = \pi/6$  and  $\vartheta = \pi/8$ .*

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